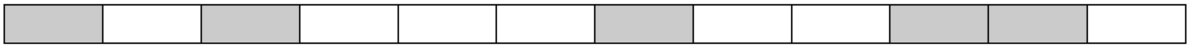
Flexibility with Ratios



From a fraction point of view we can say that $\frac{5}{12}$ of this figure is shaded. Here the denominator identifies the number of equal partitions of the whole and the numerator the number of these partitions identified. The fraction itself represents how much of the figure is shaded. Here we can say that it is less than half shaded. In fractions we are comparing a part of a whole against that whole.

When we think in ratios we can compare the number of shaded parts against the number of non-shaded parts. Either can end up in the numerator or denominator depending on how one chooses to do such a comparison. In this comparison we are comparing the parts of a whole against each other. In the above figure we can represent the ratio as follows:

$\frac{shaded}{non-shaded} = \frac{5}{7}$ or $\frac{non-shaded}{shaded} = \frac{7}{5}$

It is important that children learn to make a handy choice in how to do the comparison. It helps them later with solving proportions and identifying corresponding parts (parts with the same attribute).

<u>Consider:</u>
If for every three shaded parts there are two non-shaded parts, then how many shaded parts will there be if our figure has six non-shaded parts?

If we set our comparison up as shaded versus non-shaded we can make the following proportion:

$$\frac{3 shaded}{2 non-shaded} = \frac{?\, shaded}{6 non-shaded}$$

We solve this proportion directly by multiplying both sides by 6. There is no need for cross multiplication.

$$\frac{3}{2} = \frac{n}{6} \Rightarrow 6 \cdot \frac{3}{2} = n \Rightarrow 9 = n$$

> QUESTION: **What is the mathematical justification for cross-multiplying?** The answer lies in three important mathematical principles: 1. When you multiply both sides of an equation with the same quantity, the new equation is equivalent and will have the same solution; 2. The identity property for multiplication (the product of a number and its reciprocal is equal to 1); and 3. In a unity fraction the denominator is equal to the numerator.
> $$\frac{3}{2} = \frac{n}{6} \Rightarrow \frac{6}{n} \cdot \frac{3}{2} = \frac{n}{6} \cdot \frac{6}{n} \Rightarrow \frac{6 \cdot 3}{n \cdot 2} = 1 \Rightarrow 6 \cdot 3 = n \cdot 2$$

Or: $\frac{3}{2} = \frac{n}{6} \Rightarrow \frac{6}{6} \cdot \frac{3}{2} = \frac{n}{6} \cdot \frac{2}{2} \Rightarrow \frac{6 \cdot 3}{6 \cdot 2} = \frac{n \cdot 2}{6 \cdot 2} \Rightarrow 6 \cdot 3 = n \cdot 2$

<u>Consider:</u>
If for every three shaded parts there are two non-shaded parts, then how many non-shaded parts will there be if our figure has twelve shaded parts?

If we set our comparison up in the same way as above, we end up with our unknown quantity in the denominator. This is always unpleasant.

$$\frac{3shaded}{2non-shaded} = \frac{12shaded}{?non-shaded} \quad \text{(A)}$$

This does appear to require cross-multiplication. However, by switching the comparison we can simplify things:

$$\frac{2non-shaded}{3shaded} = \frac{?non-shaded}{12shaded} \quad \text{(B)}$$

Note that this appears to be a reflection over a horizontal line.

But I do not need to compare shaded with non-shaded. I can compare this situation also as follows:

$$\frac{?non-shaded}{2non-shaded} = \frac{12shaded}{3shaded} \quad \text{(C)}$$

Note that when you compare expression A with C, it seems that we switched "?non-shaded" with "3shaded." In situation C we also do not need to cross multiply.

Therefore, if we can set up proportions such that the unknown is in one of the numerators, we can avoid cross-multiplying and get to a solution with less steps (more efficiently).

To help students with this I have used a "proportion cross." This cross can rotate by 90, 180, and 270 degrees in either direction. This would work as follows:

3	12
2	n

Original

2	3
n	12

90 clockwise

n	2
12	3

180 clockwise

12	n
3	2

270 clockwise

The last two proportions are easier to solve.

Proportions

We can use transformation geometry (rotations and line reflections) to manipulate a proportion once set up. This allows us to create an equivalent proportion that has the unknown in the left top. When that is the case the value of the unknown is found the easiest as follows:

A. $\frac{x}{a} = \frac{b}{c} \Rightarrow x = \frac{b}{c} \times a$

Note: I multiplied both sides by a, the reciprocal of $1/a$, and a quantity times its reciprocal equals 1. No cross multiplication necessary. Only one step is required. That is why this is the simplest or root case.

So let's look at the three other cases and how we can bring these back to this root case:

B. $\frac{a}{x} = \frac{b}{c} \Rightarrow \frac{x}{a} = \frac{c}{b} \Rightarrow x = \frac{c}{b} \times a$

In this case we reflect the proportion over a horizontal axis going through the equal sign. It immediately reduces the situation back to root case.

C. $\frac{a}{b} = \frac{x}{c} \Rightarrow \frac{x}{c} = \frac{a}{b} \Rightarrow x = \frac{a}{b} \times c$

Equivalence by reflection over a vertical line through the equal sign.

D. $\frac{a}{b} = \frac{c}{x} \Rightarrow \frac{x}{b} = \frac{c}{a} \Rightarrow x = \frac{c}{a} \times b$

Equivalence by reflection over a line through b and c. This effectively switches x and a. Done!

Percents and ratios.

First a pet peeve:

When converting a fraction to a percentage, students are often taught to do the division (to create a decimal equivalent) and then multiply by 100. **This is not correct.** We know that when you multiply something by 100 it will increase 100-fold in size. The percentage should be equivalent to the fraction. As such we need to use the identity property once again. We should teach that after the division we multiply by 100%. Because 100% is equal to 1, we, in effect, multiply by 1, thus creating an equivalent expression.

Example: $\frac{3}{8} = \frac{3}{8} \times 100\% = 0.375 \times 100\% = 37.5\%$

This then leads us back to the movement of the digits when we multiply by a 100 units (See the article . Here a unit is 1% (which is located in the hundredths place of the regular base-10 system).
Generally students are presented with the following equation for solving problems with percentages:

$$\frac{part}{whole} = \frac{rate}{100}$$

Note that in any variant of the percentage problems three things can be unknown: The whole, the part of the whole we are looking for, or the rate (part of 100).

We can manipulate this proportion in a proportion cross just the same as we have done before. If the rate is unknown we multiply both sides by 100. If the part is unknown we can multiply both sides by the whole. For these two situations we can leave the proportion unchanged as above.

$$\frac{part}{whole} = \frac{rate}{100} \Rightarrow 100 \times \frac{part}{whole} = rate \qquad \frac{part}{whole} = \frac{rate}{100} \Rightarrow part = \frac{rate}{100} \times whole$$

If the whole is unknown we can re-write the proportion by reflecting it horizontally and multiply both sides by the part:

$$\frac{whole}{part} = \frac{100}{rate} \Rightarrow whole = \frac{100}{rate} \times part$$

Connection with Algebra

In algebra students study direct and indirect proportion. We denote phenomena that have direct proportion between related variable quantities (e.g. x and y) as $y = \mathrm{m}\,x$. One of the things we have students do is generate a table of values for such a relationship. Let's look at $y=2x$ for example.

x	1	2	3	4	6.5	12	?	π
y	2	4	6	8	13	24	43	?

(A)

Note that you can take any two ordered pairs and determine that their cross products are equal.
Note that students can understand the relationship between the variables both from a doubling (y = 2x) and a halving (x = ½ y) point of view. We can reflect the table horizontally and have y on top.

y	2	4	6	8	13	24	43	?
x	1	2	3	4	6.5	12	?	π

(B)

We can also represent the table vertically, which rotates the relationships 90 degrees.

(C)

x	y
1	2
2	4
3	6
4	8
6.5	13
12	24
?	43
π	?

And we can then reflect this table vertically:

(D)

y	x
2	1
4	2
6	3
8	4
13	6.5
24	12
43	?
?	π

Let’s lift out two ordered pairs (1,2) and (?,43) and relate this to the proportion crosses from the previous page:

(A)
$$\begin{array}{c|c} 1 & ? \\ \hline 2 & 43 \end{array}$$

(B)
$$\begin{array}{c|c} 2 & 43 \\ \hline 1 & ? \end{array}$$

(C)
$$\begin{array}{c|c} 1 & 2 \\ \hline ? & 43 \end{array}$$

(D)
$$\begin{array}{c|c} 2 & 1 \\ \hline 43 & ? \end{array}$$

(A) Reflect over y-axis and multiply both sides by 43
(B) Switch 2 and the unknown. Done!
(C) Reflect over x-axis. Done!
(D) Switch 2 and unknown, then multiply both sides by 43.